Field theoretical methods in condensed matter

Problem set 6

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1. Winding Numbers and the Path Integral

- For a particle moving on a ring with Hamiltonian $H = -(1/2I)(\partial^2/\partial\theta^2)$, where θ is the angle, show from first principles that the partition function $Z = \mathrm{Tr}\, e^{-\beta H}$ is given by

$$Z = \sum_{n=-\infty}^{\infty} e^{-\beta \frac{n^2}{2I}}.$$

- Now using the Feynman path integral show that the partition function can also be written as

$$Z = \int_0^{2\pi} \mathrm{d}\theta \sum_{m=-\infty}^{\infty} \int_{c_m} D\theta(\tau) e^{-\frac{I}{2}\int_0^\beta \mathrm{d}\tau \dot{\theta}^2}.$$

  The contour c_m corresponds to the integral from $\theta(0) = \theta$ to $\theta(\beta) = \theta(0) + 2\pi m$. The partition function therefore separates into contributions with different winding numbers (topological sectors). Discuss why the partition function takes this form!

- By varying the action with respect to θ show that the path integral is minimized by the classical paths $\bar{\theta}(\tau) = \theta + 2\pi m\tau/\beta$. By parameterizing a general path as $\theta(\tau) = \bar{\theta}(\tau) + \eta(\tau)$, where $\eta(\tau)$ is a path with no net winding, show that

$$Z = Z_0 \sum_{m=-\infty}^{\infty} e^{-\frac{I}{2}\frac{(2\pi m)^2}{\beta}}.$$

  Z_0 is the partition function for a free particle with open boundary conditions:

$$Z_0 = \int D\eta(\tau) e^{-\frac{I}{2}\int_0^\beta \mathrm{d}\tau (\partial_\tau \eta)^2}.$$

  Show that $Z_0 = \sqrt{I/2\pi\beta}$.

– Using Poisson's summation formula

$$\sum_{m=-\infty}^{\infty} h(m) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\phi h(\phi) e^{2\pi i n\phi}$$

show that

$$Z_0 \sum_{m=-\infty}^{\infty} e^{-\frac{I}{2}\frac{(2\pi m)^2}{\beta}} = \sum_{n=-\infty}^{\infty} e^{-\beta\frac{n^2}{2I}}$$

and thus that the two forms for the partition function are equivalent.

2. Coherent State Functional Integrals

Consider the partition function

$$Z = \mathrm{Tr} e^{-\beta(H-\mu N)}$$

for the diagonal bosonic Hamiltonian $H = \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha$, where $N = \sum_\alpha a_\alpha^\dagger a_\alpha$ is the number operator.

* Write the coherent state functional integral for the partition function in both continuous and discrete representations (with respect to imaginary time).
* Prove that

$$\prod_{k=1}^{M} \int \frac{\mathrm{d}\phi_k^* \, \mathrm{d}\phi_k}{\mathcal{N}} e^{-\sum_{j,k=1}^{M} \phi_j^* S_{jk} \phi_k} = (\mathrm{Det}\,[S_{jk}])^{-1}$$

  where $\mathcal{N}$ is some appropriate normalization. *Hint: Diagonalize the matrix S.*
* By using the previous result and calculating the resultant determinant find the partition function for this system in the continuum limit $M \to \infty$. What is the matrix S explicitly?