Field theoretical methods in condensed matter

Problem set 3

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1. Free spinless fermions

We want to consider one of the simplest models in tight-binding description: free spinless fermions which can move along a one-dimensional lattice described by the Hamiltonian

$$H = -t \sum_{j=1}^{N} [c_j^\dagger c_{j+1} + h.c.] - \mu \sum_{j=1}^{N} n_j .$$

Here t is the hopping amplitude related to the overlap of neighboring Wannier orbitals and $c_j^\dagger$ (c_j) creates (annihilates) a spinless fermion in the orbital localized around site j. $n_j = c_j^\dagger c_j$ is the particle number operator and μ the chemical potential. Physically, the fermions can be effectively spinless for various reasons. For example, they could be spin polarized by a strong external magnetic field so that all particles have the same spin. Another reason to consider this model will become clear in the lecture: Heisenberg models describing magnetic interactions can be mapped onto such spinless fermion models as well.

(a) Diagonalize the Hamiltonian by switching to a Bloch basis using a Fourier transform, see Eq. (III.A.12) of the lecture.

(b) Expand the dispersion relation ϵ_k near the Fermi points k_F, i.e., the points in momentum space with $\epsilon_{k_F} = 0$ up to third order. When does the term $\sim k^2$ vanish? What is the mathematical and physical reason for this?

(c) Calculate the temperature-dependence of the free energy $f(T)$ to leading orders in T by using (i) the exact dispersion ϵ_k and (ii) the *linearized dispersion* $\epsilon_k \approx v_F(k \pm k_F)$ as obtained in (b). What is v_F here? At some point it will be useful to perform the continuum limit $\sum_k \to (N/2\pi) \int dk$. Note that we have set the lattice constant $a = 1$ throughout. Which properties of $f(T)$ at small temperatures are universal?
[Tip: To evaluate the integrals at low temperatures one can bring them into a form such that a Sommerfeld expansion can be used.]