Field theoretical methods in condensed matter

Problem set 2

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1. **Temperature scaling**

   We want to consider a one-dimensional system which has elementary excitations with dispersion ω_k described by the non-interacting Hamiltonian

   $$H = \sum_k \hbar\omega_k (a_k^\dagger a_k + 1/2).$$

   Study the temperature-dependence of the free energy and specific heat in the low-temperature limit for

   (a) $\omega_k = \frac{\hbar k^2}{2m}$,

   (b) $\omega_k = \frac{\Delta}{\hbar} + \frac{\hbar k^2}{2m}$

   where Δ is an excitation gap which has units of energy. For (a) and (b) consider both the bosonic and the fermionic case! Discuss if the results are universal. What happens in the high-temperature limit?

2. **Schwinger bosons**

   In the Schwinger boson representation the quantum mechanical spin - which fulfills an angular momentum algebra - is represented as

   $$S^+ = a^\dagger b\,; \quad S^- = (S^+)^\dagger\,; \quad S^z = \frac{1}{2}(a^\dagger a - b^\dagger b)$$

   where a, b are bosonic operators.

   (a) Show that this relation is consistent with the commutation relation $[S^+, S^-] = 2S^z$.

   (b) Using the bosonic commutation relations show that

   $$|S, m\rangle = \frac{(a^\dagger)^{S+m}}{\sqrt{(S+m)!}} \frac{(b^\dagger)^{S-m}}{\sqrt{(S-m)!}} |0\rangle$$

   is compatible with the definition of an eigenstate of the total spin operator $\mathbf{S}^2$ and S^z. Here $|0\rangle$ denotes the vacuum of the Schwinger bosons and the total spin S defines the physical subspace $\{|n_a, n_b\rangle \mid n_a + n_b = 2S\}$ where $n_{a,b}$ are the occupation numbers.